

CHARACTERISTIC SPHERES OF FREE DIFFERENTIABLE ACTIONS OF S^1 AND S^3 ON HOMOTOPY SPHERES

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1. Introduction. Let (S^i, Σ^m, φ) , $i=0, 1$ or 3 , be a free differentiable action of S^i on the homotopy m -sphere Σ^m with orbit space Σ^m/S^i and a homotopy equivalence $f: \Sigma^m/S^i \rightarrow KP^k$ (K =real, complex or quaternionic according to whether $i=0, 1$ or 3). By a *characteristic homotopy $(m-q)$ -sphere* of Σ^m we mean a homotopy sphere Σ^{m-q} which is S^i -invariant, that is $\varphi(S^i \times \Sigma^{m-q}) \subset \Sigma^{m-q} \subset \Sigma^m$, and such that f is transverse regular on KP^i with $f^{-1}(KP^i) = \Sigma^{m-q}/S^i$ and $f|_{\Sigma^{m-q}/S^i}$ is a homotopy equivalence. In this paper, we are concerned with the problem of finding characteristic homotopy spheres. The case $i=0$ and $q=1$ was studied by Browder and Livesay [3]. For the case $i=1$ and $q=2$, Montgomery and Yang have shown that the obstruction is precisely the Browder-Livesay invariant which is obtained by restricting the action to the subgroup Z_2 [11]. We consider the case $i=3$, $m=4n+3$, and compare the obstructions between the S^3 -action and S^1 -action for $S^1 \subset S^3$. We also give some interesting examples in dimensions 11, 13 and 15. Our methods will be based on the computation of the surgery obstruction by using a formula of Browder [2, 4.4].

Throughout the paper, Z denotes the ring of integers, $\sigma(M)$ the index of the smooth manifold M , $\tau(M)$ the tangent bundle of M . We let CP^n be the complex projective n -space and QP^n be the quaternion projective n -space.

2. The invariants $I_{2k}(S^1, \Sigma^{2n+1})$ and $I_{4k}(S^3, \Sigma^{4n+3})$. Suppose that S^1 (resp. S^3) acts freely and differentiably on a homotopy sphere Σ^{2n+1} (resp. Σ^{4n+3}), and let $N = \Sigma^{2n+1}/S^1$ (resp. Σ^{4n+3}/S^3) be the orbit space. Let $f: N \rightarrow CP^n$ (resp. $f: N \rightarrow QP^n$) be a homotopy equivalence which is transverse regular on the submanifold CP^{n-k} (resp. QP^{n-k}) with $n-k > 2$ (resp. $n-k > 1$), and let $M = f^{-1}(CP^{n-k})$ (resp. $f^{-1}(QP^{n-k})$). Furthermore we assume that $\dim M = 4q$. There is an obstruction to make f normally cobordant to $f': N \rightarrow CP^n$ (resp. $f': N \rightarrow QP^n$) a homotopy equivalence, such that if $M' = (f')^{-1}(CP^{n-k})$ (resp. $(f')^{-1}(QP^{n-k})$), $f': (N, M') \rightarrow (CP^n, CP^{n-k})$ (resp. $f': (N, M') \rightarrow (QP^n, QP^{n-k})$) is a homotopy equivalence on each term [2, 2.14]. The obstruction is simply the difference of two indices, namely $\sigma(M) - \sigma(CP^{n-k})$ (resp. $\sigma(M) - \sigma(QP^{n-k})$) which lies in the group $8Z$, and we shall denote it by $I_{2k}(S^1, \Sigma^{2n+1})$ (resp. $I_{4k}(S^3, \Sigma^{4n+3})$). It is precisely the obstruction of the free S^1 (resp. S^3) action on Σ^{2n+1} (resp. Σ^{4n+3}) having codimension $2k$ (resp. $4k$)

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characteristic homotopy sphere. For S^1 action with codimension 2 characteristic sphere, the obstruction is the invariant $I(S^1, \Sigma^{2n+1})$ defined by Montgomery and Yang [11], hence $I_2(S^1, \Sigma^{2n+1}) = I(S^1, \Sigma^{2n+1})$ if $n \geq 4$. We restate a result of Browder in [2, 4.4] as the following theorem:

THEOREM 2.1 [2]. *Let (S^1, Σ^{2n+1}) or (S^3, Σ^{4n+3}) be a free differentiable action. Then*

$$\begin{aligned} I_{2k}(S^1, \Sigma^{2n+1}) &= \langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k} \rangle - 1, \\ I_{4k}(S^3, \Sigma^{4n+3}) &= \langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k} \rangle - \sigma(QP^{n-k}), \end{aligned}$$

where ρ is the canonical bundle over $N = \Sigma^{2n+1}/S^1$ or Σ^{4n+3}/S^3 , associated to the principal bundle

$$S^1 \rightarrow \Sigma^{2n+1} \rightarrow \Sigma^{2n+1}/S^1 \quad \text{or} \quad S^3 \rightarrow \Sigma^{4n+3} \rightarrow \Sigma^{4n+3}/S^3$$

respectively, ρ^{-1} the inverse of ρ , χ_{n-k} is the generator of $H_{4q}(N)$, $n-k=2q$ or $n-k=q$, and L_q the q th component of the Hirzebruch's L -genus [5],

$$L: (KO)^\sim(N) \rightarrow \sum_{i \geq 0} H^{4i}(N, \mathbb{Q}).$$

DEFINITION 2.2. Let $p_i(CP^n, Z) = r_i \bar{\alpha}^{2i}$ (resp. $p_i(QP^n, Z) = \bar{r}_i \bar{\beta}^i$). A homotopy complex (resp. quaternion) projective n -space M is called *semistandard* if the Pontrjagin classes $p_i(M) = r_i \alpha^{2i}$ for $i \leq [(n-1)/4]$ (resp. $p_i(M) = \bar{r}_i \beta^i$, $i \leq [n/2]$), where $\alpha, \bar{\alpha}$ (resp. $\beta, \bar{\beta}$) are the generators of $H^2(M, Z)$, $H^2(CP^n, Z)$ (resp. $H^4(M, Z)$, $H^4(QP^n, Z)$) respectively.

Suppose that a free differentiable S^3 -action on a homotopy sphere Σ^{4n+3} is given ($n \geq 3$) such that the orbit space Σ^{4n+3}/S^3 is a semistandard quaternion projective space. If we restrict the action to the subgroup S^1 of S^3 , we have the fibre bundle $\eta: S^2 \rightarrow \Sigma^{4n+3}/S^1 \xrightarrow{\pi} \Sigma^{4n+3}/S^3$ with $p(\eta) = 1 + 4\bar{\alpha}$, $\bar{\alpha}$ the generator of $H^4(\Sigma^{4n+3}/S^3, Z)$ and $p(QP^n) = (1 + \alpha)^{2(n+1)}(1 + 4\alpha)^{-1}$, α the generator of $H^4(QP^n, Z)$ [1]. Hence we see that Σ^{4n+3}/S^1 is also semistandard. In [6], W. C. Hsiang has shown the existence of infinitely many nonhomeomorphic semistandard complex $(2n+1)$ -spaces and quaternion n -spaces such that their Pontrjagin classes p_i are distinct for $i \geq [n/2] + 1$.

THEOREM 2.3. *Suppose that S^3 acts freely and differentiably on a homotopy $(4n+3)$ -sphere Σ^{4n+3} ($n \geq 3$) such that the orbit space Σ^{4n+3}/S^3 is a semistandard homotopy quaternion projective space. Then*

$$\begin{aligned} &I_{4n-4[n/2]-4j}(S^3, \Sigma^{4n+3}) \\ (1) \quad &= \sum_{i=0}^{j-1} \{ \langle L_{[n/2]+j-i}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+j-i} \rangle - \langle L_{[n/2]+j-i}(\tau(QP^n)), \bar{\chi}_{[n/2]+j-i} \rangle \} \\ &\quad \times \langle L_i((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_i \rangle \quad \text{for } j \geq 1, \end{aligned}$$

$$\begin{aligned}
 & I_{4n-4[n/2]-4j+2}(S^1, \Sigma^{4n+3}) \\
 (2) \quad & = \sum_{i=0}^{j-1} \{ \langle L_{[n/2]+j-i}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+j-i} \rangle - \langle L_{[n/2]+j-i}(\tau(QP^n)), \bar{\chi}_{[n/2]+j-i} \rangle \} \\
 & \quad \times \langle L_i(\pi^* \eta \oplus (2n-2[n/2]-2j+1)\rho^{-1}), \chi_i \rangle \quad \text{for } j \geq 1, \\
 & I_{4n-4[n/2]-4}(S^3, \Sigma^{4n+3}) = I_{4n-4[n/2]-2}(S^1, \Sigma^{4n+3}) \\
 (3) \quad & = s_{[n/2]+1} \{ \langle p_{[n/2]+1}(\Sigma^{4n+3}/S^3), \bar{\chi}'_{[n/2]+1} \rangle \\
 & \quad - \langle p_{[n/2]+1}(QP^n), \bar{\chi}_{[n/2]+1} \rangle \}, \\
 & I_{4n-4[n/2]+4j}(S^3, \Sigma^{4n+3}) \\
 (4) \quad & = I_{4n-4[n/2]+4j+2}(S^1, \Sigma^{4n+3}) = 0 \quad \text{for } [n/2]-2 \geq j \geq 0,
 \end{aligned}$$

where $\rho, \bar{\rho}$ denote the canonical bundles over CP^{2n+1} and QP^n , $\chi_d, \bar{\chi}_d$ and $\bar{\chi}'_d$ the generators of $H_{4d}(CP^{2n+1}, Z)$, $H_{4d}(QP^n, Z)$ and $H_{4d}(\Sigma^{4n+3}/S^3, Z)$, respectively (cf. Theorem 2.1), $s_{[n/2]+1}$ the coefficient of $\rho_{[n/2]+1}$ in $L_{[n/2]+1}$ [5, p. 12], and $[n/2]$ is the largest integer less than or equal to $n/2$.

Notation. In the proof, we will denote the canonical bundles over Σ^{4n+3}/S^1 and Σ^{4n+3}/S^3 by ρ' and $\bar{\rho}'$ respectively, and χ'_d the generator of $H_{4d}(\Sigma^{4n+3}/S^1, Z)$. Let $\tau = \tau(CP^{2n+1})$, $\tau' = \tau(\Sigma^{4n+3}/S^1)$, $\bar{\tau} = \tau(QP^n)$ and $\bar{\tau}' = \tau(\Sigma^{4n+3}/S^3)$.

Proof. Since Σ^{4n+3} admits free S^3 -action, we have fibre bundle

$$\eta': S^2 \longrightarrow \Sigma^{4n+3}/S^1 \xrightarrow{\pi'} \Sigma^{4n+3}/S^3$$

which is homotopically equivalent to the standard fibration $\eta: S^2 \rightarrow CP^{2n+1} \xrightarrow{\pi} QP^n$ by $(f, \bar{f})$, that is, we have the following commutative diagram:

$$\begin{array}{ccc}
 \eta': S^2 & \longrightarrow & \Sigma^{4n+3}/S^1 \xrightarrow{\pi'} \Sigma^{4n+3}/S^3 \\
 & & \downarrow f \qquad \qquad \downarrow \bar{f} \\
 \eta: S^2 & \longrightarrow & CP^{2n+1} \xrightarrow{\pi} QP^n
 \end{array}$$

Notice that $\rho' = f^*(\rho)$, $\bar{\rho}' = \bar{f}^*(\bar{\rho})$ and $\eta' = \bar{f}^*(\eta)$. By assumption we have

$$\begin{aligned}
 (5) \quad & \bar{f}^*(p_i(QP^n)) = p_i(\Sigma^{4n+3}/S^3) \quad \text{and} \\
 & f^*(p_i(CP^{2n+1})) = p_i(\Sigma^{4n+3}/S^1) \quad \text{for } i \leq [n/2].
 \end{aligned}$$

Now we consider the standard free action of S^3 on S^{4n+3} . This action has characteristic $(4[n/2]+4j+3)$ -sphere $S^{4[n/2]+4j+3}$. Hence by Theorem 2.1, we have

$$\langle L_{[n/2]+j}(\bar{\tau} \oplus (n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_{[n/2]+j} \rangle = \sigma(QP^{[n/2]+j})$$

or

$$\begin{aligned}
 (6) \quad & \langle L_{[n/2]}(\bar{\tau})L_j((n-[n/2]-j)\bar{\rho}^{-1}) + \cdots + L_{[n/2]+j}((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_{[n/2]+j} \rangle \\
 & = \sigma(QP^{[n/2]+j}) - \langle L_{[n/2]+j}(\bar{\tau}) + \cdots + L_{[n/2]+1}(\bar{\tau}) \\
 & \quad \times L_{j-1}((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_{[n/2]+j} \rangle.
 \end{aligned}$$

Since $\tilde{f}^*(p_i(QP^n)) = p_i(\Sigma^{4n+3}/S^3)$ for $i \leq [n/2]$, $L_i(\tilde{\tau}') = \tilde{f}^* L_i(\tilde{\tau})$ for $i \leq [n/2]$. Thus we obtain

$$\begin{aligned}
 & \langle L_{[n/2]}(\tilde{\tau}') L_j((n - [n/2] - j)(\tilde{\rho}')^{-1}) + \cdots + L_{[n/2]+j}((n - [n/2] - j)(\tilde{\rho}')^{-1}), \tilde{\chi}'_{[n/2]+j} \rangle \\
 &= \langle \tilde{f}^* \{ L_{[n/2]}(\tilde{\tau}) L_j((n - [n/2] - j)\tilde{\rho}^{-1}) \\
 (7) \quad & \quad \quad \quad + \cdots + L_{[n/2]+j}((n - [n/2] - j)\tilde{\rho}^{-1}) \}, \tilde{\chi}'_{[n/2]+j} \rangle \\
 &= \sigma(QP^{[n/2]+j}) - \langle L_{[n/2]+j}(\tilde{\tau}) + \cdots + L_{[n/2]+1}(\tilde{\tau}) \\
 & \quad \quad \quad \times L_{j-1}((n - [n/2] - j)\tilde{\rho}^{-1}), \tilde{\chi}_{[n/2]+j} \rangle
 \end{aligned}$$

by (6). Again by Theorem 2.1,

$$\begin{aligned}
 & I_{4n-4[n/2]-4j}(S^3, \Sigma^{4n+3}) \\
 &= \langle L_{[n/2]+j}(\tilde{\tau}') + \cdots + L_{[n/2]+1}(\tilde{\tau}') L_{j-1}((n - [n/2] - j)(\tilde{\rho}')^{-1}), \tilde{\chi}'_{[n/2]+j} \rangle \\
 & \quad + \langle L_{[n/2]}(\tilde{\tau}') L_j((n - [n/2] - j)(\tilde{\rho}')^{-1}) \\
 & \quad \quad \quad + \cdots + L_{[n/2]+j}((n - [n/2] - j)(\tilde{\rho}')^{-1}), \tilde{\chi}'_{[n/2]+j} \rangle - \sigma(QP^{[n/2]+j}) \\
 &= \sum_{i=0}^{j-1} \{ \langle L_{[n/2]+j-i}(\tau(\Sigma^{4n+3}/S^3)), \tilde{\chi}'_{[n/2]+j-i} \rangle \\
 & \quad \quad \quad - \langle L_{[n/2]+j-i}(\tau(QP^n)), \tilde{\chi}_{[n/2]+j-i} \rangle \} \langle L_i((n - [n/2] - j)\tilde{\rho}^{-1}), \tilde{\chi}_i \rangle
 \end{aligned}$$

by (7), and (1) is proved.

The proof of (2) is similar. We have $\tau' = \pi'^* \tilde{\tau}' \oplus \pi'^* \eta'$ and $\tau = \pi^* \tilde{\tau} \oplus \pi^* \eta$. As we remarked before, $f^* p_i(\tau) = p_i(\tau')$ for $i \leq [n/2]$. The standard free S^1 -action on S^{4n+3} has characteristic sphere $S^{4[n/2]+4j+1}$, hence

$$\begin{aligned}
 1 &= \langle L_{[n/2]+j}(\tau \oplus (2n - 2[n/2] - 2j + 1)\rho^{-1}), \chi_{[n/2]+j} \rangle \\
 &= \langle L_{[n/2]+j}(\pi^* \tilde{\tau} \oplus \pi^* \eta \oplus (2n - 2[n/2] - 2j + 1)\rho^{-1}), \chi_{[n/2]+j} \rangle.
 \end{aligned}$$

For simplicity, let $k = 2n - 2[n/2] - 2j + 1$, then

$$\begin{aligned}
 & \langle L_{[n/2]}(\pi^* \tilde{\tau}) L_j(\pi^* \eta \oplus k\rho^{-1}) + \cdots + L_{[n/2]+j}(\pi^* \eta \oplus k\rho^{-1}), \chi_{[n/2]+j} \rangle \\
 &= 1 - \langle L_{[n/2]+j}(\pi^* \tilde{\tau}) + \cdots + L_{[n/2]+1}(\pi^* \tilde{\tau}) L_{j-1}(\pi^* \eta \oplus k\rho^{-1}), \chi_{[n/2]+j} \rangle \\
 (8) \quad &= 1 - \langle L_{[n/2]+j}(\pi^* \tilde{\tau}), \chi_{[n/2]+j} \rangle \\
 & \quad - \cdots - \langle L_{[n/2]+1}(\pi^* \tilde{\tau}), \chi_{[n/2]+1} \rangle \langle L_{j-1}(\pi^* \eta \oplus k\rho^{-1}), \chi_{j-1} \rangle \\
 &= 1 - \langle L_{[n/2]+j}(\tilde{\tau}), \tilde{\chi}_{[n/2]+j} \rangle \\
 & \quad - \cdots - \langle L_{[n/2]+1}(\tilde{\tau}), \tilde{\chi}_{[n/2]+1} \rangle \langle L_{j-1}(\pi^* \eta \oplus k\rho^{-1}), \chi_{j-1} \rangle.
 \end{aligned}$$

If we use the fact that $\pi'^* \tilde{f}^* = f^* \pi^*$, we can see that

$$\begin{aligned}
 & \langle L_{[n/2]}(\pi'^* \tilde{\tau}') L_j(\pi'^* \eta' \oplus k\rho'^{-1}) + \cdots + L_{[n/2]+j}(\pi'^* \eta' \oplus k\rho'^{-1}), \chi'_{[n/2]+j} \rangle \\
 (9) \quad &= 1 - \langle L_{[n/2]+j}(\tilde{\tau}), \tilde{\chi}_{[n/2]+j} \rangle \\
 & \quad - \cdots - \langle L_{[n/2]+1}(\tilde{\tau}), \tilde{\chi}_{[n/2]+1} \rangle \langle L_{j-1}(\pi^* \eta \oplus k\rho^{-1}), \chi_{j-1} \rangle.
 \end{aligned}$$

Therefore, if we substitute (9) into the formula for $I_{4n-4[n/2]-4j+2}(S^1, \Sigma^{4n+3})$, the conclusion of (2) follows. The proof of (4) is essentially the same but a little

simpler. The statement (3) follows from (1), (2) and (5). This completes the proof of Theorem 2.3.

As a simple consequence, we state the following:

COROLLARY 2.4. (i) *If $I_{4n-4[n/2]-4}(S^3, \Sigma^{4n+3}) \neq 0$, then*

$$I_{4n-4[n/2]-8}(S^3, \Sigma^{4n+3}) \neq I_{4n-4[n/2]-6}(S^1, \Sigma^{4n+3}).$$

(ii) $I_2(S^1, \Sigma^{15}) = I_6(S^1, \Sigma^{15})$ and $I_2(S^1, \Sigma^{19}) = I_6(S^1, \Sigma^{19})$, since $L_3(\Sigma^{15}/S^3) = 0$, $L_4(\Sigma^{19}/S^3) = 1$ and $\langle L_1(\pi^*\eta \oplus \rho^{-1}), \chi_1 \rangle = 1$.

(iii) *For $n \geq 5$, $I_{4n-4[n/2]-2}(S^1, \Sigma^{4n+3}) = I_{4n-4[n/2]-6}(S^1, \Sigma^{4n+3})$ if and only if*

$$\begin{aligned} & 3\{\langle L_{[n/2]+2}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+2} \rangle - \langle L_{[n/2]+2}(\tau(QP^n)), \bar{\chi}_{[n/2]+2} \rangle\} \\ &= (2n-2[n/2]-4)\{\langle L_{[n/2]+1}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+1} \rangle \\ &\quad - \langle L_{[n/2]+1}(\tau(QP^n)), \bar{\chi}_{[n/2]+1} \rangle\}. \end{aligned}$$

We may derive the similar results in other cases.

To conclude this section we prove the following characterization theorem.

THEOREM 2.5. *Let S^3 act freely and differentiably on a homotopy $(4n+3)$ -sphere Σ^{4n+3} ($n \geq 4$). Then the orbit space Σ^{4n+3}/S^3 is a semistandard homotopy quaternion projective space if and only if (4) is satisfied.*

Proof. We use the notation of Theorem 2.3. Suppose (4) holds. It suffices to show that the following relations are satisfied:

$$(10) \quad \langle L_j(\bar{\tau}'), \bar{\chi}'_j \rangle = \langle L_j(\bar{\tau}), \bar{\chi}_j \rangle \quad \text{for } 1 \leq j \leq [n/2].$$

From $I_{4n-8}(S^3, \Sigma^{4n+3}) = I_{4n-6}(S^1, \Sigma^{4n+3}) = 0$, we obtain

$$\begin{aligned} \langle L_2(\bar{\tau}' \oplus (n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_2 \rangle &= \langle L_2(\tau' \oplus (2n-3)(\rho')^{-1}), \chi'_2 \rangle \\ &= \langle L_2(\pi'^*\bar{\tau}' \oplus \pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_2 \rangle \end{aligned}$$

because $\tau' = \pi'^*\bar{\tau}' \oplus \pi'^*\eta'$. Simplifying this equation we get

$$(11) \quad \begin{aligned} \langle L_1(\bar{\tau}'), \bar{\chi}'_1 \rangle \{ \langle L_1((n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_1 \rangle - \langle L_1(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_1 \rangle \} \\ = \langle L_2(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_2 \rangle - \langle L_2((n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_2 \rangle. \end{aligned}$$

Similarly if we consider the standard free S^3 action on S^{4n+3} , we have

$$I_{4n-8}(S^3, S^{4n+3}) = I_{4n-6}(S^1, S^{4n+3}) = 0;$$

thus the same argument implies

$$(12) \quad \begin{aligned} \langle L_1(\bar{\tau}), \bar{\chi}_1 \rangle \{ \langle L_1((n-2)(\bar{\rho})^{-1}), \bar{\chi}_1 \rangle - \langle L_1(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_1 \rangle \} \\ = \langle L_2(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_2 \rangle - \langle L_2((n-2)(\bar{\rho})^{-1}), \bar{\chi}_2 \rangle. \end{aligned}$$

By comparing (11) and (12) we have

$$(13) \quad \langle L_1(\bar{\tau}'), \bar{\chi}'_1 \rangle = \langle L_1(\bar{\tau}), \bar{\chi}_1 \rangle$$

since

$$\begin{aligned}\langle L_1((n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_1 \rangle &= \langle L_1((n-2)\bar{\rho}^{-1}), \bar{\chi}_1 \rangle, \\ \langle L_1(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_1 \rangle &= \langle L_1(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_1 \rangle, \\ \langle L_2(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_2 \rangle &= \langle L_2(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_2 \rangle,\end{aligned}$$

and

$$\langle L_2((n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_2 \rangle = \langle L_2((n-2)\bar{\rho}^{-1}), \bar{\chi}_2 \rangle.$$

Next we see from $I_{4n-8}(S^3, \Sigma^{4n+3}) = \langle L_2(\bar{\tau}' \oplus (n-2)\bar{\rho}^{-1}), \bar{\chi}'_2 \rangle - 1 = 0$ and $I_{4n-8}(S^3, S^{4n+3}) = 0$ that

$$(14) \quad \langle L_2(\bar{\tau}'), \bar{\chi}'_2 \rangle = \langle L_2(\bar{\tau}), \bar{\chi}_2 \rangle.$$

Now we suppose that

$$\langle L_i(\bar{\tau}'), \bar{\chi}'_i \rangle = \langle L_i(\bar{\tau}), \bar{\chi}_i \rangle \quad \text{for } 1 \leq i < [n/2].$$

But

$$I_{4n-4i-4}(S^3, \Sigma^{4n+3}) = \langle L_{i+1}(\bar{\tau}' \oplus (n-i-1)(\bar{\rho}')^{-1}), \bar{\chi}'_{i+1} \rangle - \sigma(QP^{i+1}) = 0$$

and

$$I_{4n-4i-4}(S^3, S^{4n+3}) = \langle L_{i+1}(\bar{\tau} \oplus (n-i-1)\bar{\rho}^{-1}), \bar{\chi}_{i+1} \rangle - \sigma(QP^{i+1}) = 0.$$

Thus,

$$\begin{aligned}\langle L_{i+1}(\bar{\tau}'), \bar{\chi}'_{i+1} \rangle &= \sigma(QP^{i+1}) - \langle L_i(\bar{\tau}'), \bar{\chi}'_i \rangle \langle L_1((n-i-1)(\bar{\rho}')^{-1}), \bar{\chi}'_1 \rangle \\ &\quad - \cdots - \langle L_{i+1}((n-i-1)(\bar{\rho}')^{-1}), \bar{\chi}'_{i+1} \rangle \\ &= \sigma(QP^{i+1}) - \langle L_i(\bar{\tau}), \bar{\chi}_i \rangle \langle L_1((n-i-1)\bar{\rho}^{-1}), \bar{\chi}_1 \rangle \\ &\quad - \cdots - \langle L_{i+1}((n-i-1)\bar{\rho}^{-1}), \bar{\chi}_{i+1} \rangle \\ &= \langle L_{i+1}(\bar{\tau}), \bar{\chi}_{i+1} \rangle.\end{aligned}$$

This completes the proof of the theorem.

3. Lower dimensional examples. This section contains some results on the free differentiable actions of Z_2 , S^1 and S^3 on homotopy spheres of dimensions 11, 13 and 15.

I. Actions on homotopy 11-spheres. In [8] we studied the differentiable actions of S^3 on homotopy 11-spheres and proved

THEOREM 3.1. *Let Σ_M^{11} denote the Milnor sphere which represents the generator of θ_{11} [9]. Then a homotopy 11-sphere Σ^{11} admits a free differentiable S^3 -action if and only if $\Sigma^{11} \approx 32k\Sigma_M^{11}$ for some $k \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16 \pmod{31}$. These all admit infinitely many topologically distinct actions which can be distinguished by the first Pontrjagin classes of the orbit spaces.*

The proof of this theorem is based on the examples constructed by the Hsiang brothers [7] and the following fact [8]: Let S^3 act freely and differentially on a homotopy 11-sphere Σ^{11} . Then either

(i) $p_1(\Sigma^{11}/S^3) = (672k+2)\bar{\alpha}$ for some integer k , and

$$\mu(\Sigma^{11}) \equiv -(13k+1)(k+1)k/31 \pmod{1},$$

or

(ii) $p_1(\Sigma^{11}/S^3) = (672k+194)\bar{\alpha}$ for some integer k , and

$$\mu(\Sigma^{11}/S^3) \equiv -7(-k+1)(7k+2)(k-7)/31 \pmod{1},$$

where $\bar{\alpha}$ denotes the generator of $H^4(\Sigma^{11}/S^3, Z)$ and $\mu(\Sigma^{11})$ the Eells-Kuiper μ -invariant [4].

It is said in [8] that the homotopy sphere $k\Sigma_M^{11}$, k odd, admits no free differentiable S^1 -actions. Since we made some errors in computation, this does not follow from Lemma 2.1 of [8]. We conjecture that this result remains true. We correct this by proving the following Lemma 3.2. The proof being essentially the same, we only sketch the proof.

LEMMA 3.2. *Let S^1 act freely and differentiably on a homotopy 11-sphere Σ^{11} . Then*

$$\mu(\Sigma^{11}) \equiv (9i + 102i^2 + 288i^3 + 148j + 720ij + 496t)/992 \pmod{1},$$

for some integers i, j and t . Moreover,

$$p_1(\Sigma^{11}/S^1) = (24i+6)\alpha^2,$$

$$p_2(\Sigma^{11}/S^1) = (1008i^2 + 264i + 15 + 1440j)\alpha^4,$$

where α denotes the generator of $H^2(\Sigma^{11}/S^1, Z)$.

Proof. Let $D^2 \rightarrow W \xrightarrow{\pi} \Sigma^{11}/S^1$ be the associated disk bundle of the principal bundle $\xi: S^1 \rightarrow \Sigma^{11} \rightarrow \Sigma^{11}/S^1$. We see that $p(\xi) = 1 + \alpha^2$, $\omega_2(W) \neq 0$ and the index $\sigma(W) = 1$. Let

$$\begin{aligned} p_1(\Sigma^{11}/S^1) &= r_1\alpha^2, & p_2(\Sigma^{11}/S^1) &= r_2\alpha^4, \\ p_1(W) &= r_3\beta^2, & p_2(W) &= r_4\beta^4. \end{aligned}$$

$\beta = \pi^*\alpha$ and $r_1, r_2, r_3, r_4 \in Z$. We have $r_3 = r_1 + 1$, $r_4 = r_1 + r_2$. Since β reduction mod 2 is $\omega_2(W)$, the invariant $\nu(\Sigma^{11})$ is well defined [10]. By substituting the above data into the formula for ν , and using the fact that $\nu(\Sigma^{11}) \equiv 2\mu(\Sigma^{11}) \pmod{1}$ [10], we have

$$\begin{aligned} 496\nu(\Sigma^{11}) &\equiv 992\mu(\Sigma^{11}) \\ &\equiv \{-312 + 80r_1 - 64r_2 + 142r_1^2 + 60r_1r_2 - 45r_1^3\}/2^6 \cdot 3^2 \cdot 5 \pmod{496}. \end{aligned}$$

The rest of the proof is just repeating the same argument as in the proof of Lemma 2.1 of [8].

THEOREM 3.3. *Let S^1 act freely and differentiably on a homotopy 11-sphere Σ^{11} . Then $I_2(S^1, \Sigma^{11}) = 16(9i^2 + 2i + 14j)$ for some integers i and j . Hence if $I_2(S^1, \Sigma^{11}) = 0$, then i is even and $\Sigma^{11} \approx m\Sigma_M^{11}$ for some even integers m .*

Proof. It is known that $p(\rho^{-1}) = (1 + \alpha^2)^{-1}$. We apply Theorem 2.1 to obtain

$$\begin{aligned} I_2(S^1, \Sigma^{11}) &= \langle L_2(\tau(\Sigma^{11}/S^1) \oplus \rho^{-1}), \chi_4 \rangle - 1 \\ &= \langle L_2(\Sigma^{11}/S^1) + L_1(\Sigma^{11}/S^1)L_1(\rho^{-1}) + L_2(\rho^{-1}), \chi_4 \rangle - 1 \\ &= \frac{1}{45} [7(1008i^2 + 264i + 15 + 1440j) - (24i + 6)^2] \\ &\quad - \frac{1}{3}(24i + 6) \cdot \frac{1}{3} + \frac{1}{45}(7 - 1) - 1 \\ &= 16(9i^2 + 2i + 14j). \end{aligned}$$

COROLLARY 3.4. *There exists free differentiable Z_2 -action on $k\Sigma_M^{11}$ for some k such that this action cannot be extended to the free differentiable action of S^1 .*

Proof. Let $I(Z_2, \Sigma^{11})$ be the Browder-Livesay invariant [3], [11]. Montgomery and Yang showed that $I(Z_2, \Sigma^{11}) = I_2(S^1, \Sigma^{11})$ [11]. Santiago has constructed the free involution (Z_2, Σ_k^{11}) for all $k \in \mathbb{Z}$ such that

$$I(Z_2, \Sigma_k^{11}) = \sigma(W_k) = 8k,$$

with Σ_k^{11} bounding a π -manifold W_k^{12} [12], [14]. Thus for k odd, or k is not of the form $2(9i^2 + 2i + 14j)$, the free Z_2 -action on $k\Sigma_M^{11}$ cannot be extended to free S^1 -action.

THEOREM 3.5. *Let S^3 act freely on homotopy 11-spheres Σ^{11} and $Z_2 \subset S^1$ are subgroups of S^3 . Then*

(i) $p_1(\Sigma^{11}/S^3) = (672k + 2)\bar{\alpha}$ for some integer k , and $I(Z_2, \Sigma^{11}) = I_2(S^1, \Sigma^{11}) = 224k$ or

(ii) $p_1(\Sigma^{11}/S^3) = (672k + 194)\bar{\alpha}$ for some integer k , and $I(Z_2, \Sigma^{11}) = I_2(S^1, \Sigma^{11}) = 224k + 64$.

In particular, Σ^{11} has S^1 -invariant characteristic 9-sphere S^9 if and only if $\Sigma^{11} \approx S^{11}$.

The involutions in (i) for different k are all differentiably distinct. This answers a question of the Hsiangs [7].

Proof. If $p_1(\Sigma^{11}/S^3) = (672k + 2)\bar{\alpha}$, then

$$p_1(\Sigma^{11}/S^1) = (672k + 6)\beta^2, \quad \text{and} \quad p_2(\Sigma^{11}/S^1) = (64512k^2 + 3072k + 15)\beta^4,$$

where $\beta^2 = \pi^*\bar{\alpha}$, $\pi: \Sigma^{11}/S^1 \rightarrow \Sigma^{11}/S^3$ the natural projection [7]. Similarly, if $p_1(\Sigma^{11}/S^3) = (672k + 194)\bar{\alpha}$, then

$$p_1(\Sigma^{11}/S^1) = (672k + 198)\beta^2, \quad \text{and} \quad p_2(\Sigma^{11}/S^1) = (6412k^2 + 39936k + 6159)\beta^4.$$

Thus (i) and (ii) are easily computed as in the proof of Theorem 3.3. We note that $I_2(S^1, \Sigma^{11}) = 0$ if and only if $p_1(\Sigma^{11}/S^3) = (672k + 2)\bar{\alpha}$ for $k = 0$. Hence $\mu(\Sigma^{11}) \equiv -(13k + 1)(k + 1)k/31 \equiv 0 \pmod{1}$, and so $\Sigma^{11} \approx S^{11}$ by [4].

II. Actions on homotopy 13-spheres. Let S^1 act on a homotopy 13-sphere Σ^{13} with $p_1(\Sigma^{13}/S^1) = 7\alpha^2$, α the generator of $H^2(\Sigma^{13}/S^1, \mathbb{Z})$. W. C. Hsiang has shown

that there are infinitely many nonequivalent actions of S^1 on some Σ^{13} with $p_1(\Sigma^{13}/S^1)=7\alpha^2$ and different second and third Pontrjagin classes [6]. Let $p_2(\Sigma^{13}/S^1)=r_2\alpha^4$ and $p_3(\Sigma^{13}/S^1)=r_3\alpha^6$. Since the index $\sigma(\Sigma^{13}/S^1)=1$, we have

$$\begin{aligned} 1 &= \langle L_3(\Sigma^{13}/S^1), [\Sigma^{13}/S^1] \rangle \\ &= \left\langle \frac{1}{3^3 \cdot 5 \cdot 7} (62p_3(\Sigma^{13}/S^1) - 13p_2(\Sigma^{13}/S^1)p_1(\Sigma^{13}/S^1) + p_1(\Sigma^{13}/S^1)^3), [\Sigma^{13}/S^1] \right\rangle \\ &= \frac{1}{3^3 \cdot 5 \cdot 7} (62r_3 - 91r_2 + 686) \end{aligned}$$

where $[\Sigma^{13}/S^1]$ denotes the fundamental class of Σ^{13}/S^1 . It is not difficult to see that

$$r_2 = 62\bar{i} + 21 \quad \text{and} \quad r_3 = 91\bar{i} + 35 \quad \text{for some } \bar{i} \in \mathbb{Z}.$$

Now α reduction modulo 2 is the second Stiefel-Whitney class $\omega_2(\Sigma^{13}/S^1)$ which is different from zero. Thus by [5], $A(\Sigma^{13}/S^1, \alpha/2)$ is an integer. A simple calculation shows that

$$A(\Sigma^{13}/S^1, \alpha/2) = \frac{-259 + 14r_2 - r_3}{2^6 \cdot 3^3 \cdot 5 \cdot 7} = \frac{37\bar{i}}{2^6 \cdot 3^3 \cdot 5}$$

hence $\bar{i} = 2^6 \cdot 3^3 \cdot 5 \cdot i$ for some $i \in \mathbb{Z}$. Thus,

$$\begin{aligned} I_4(S^1, \Sigma^{13}) &= \langle L_2(\tau(\Sigma^{13}/S^1) \oplus 2\rho^{-1}), \chi_4 \rangle - 1 \\ &= \frac{7 \cdot 62\bar{i}}{45} = 2^7 \cdot 7 \cdot 31i. \end{aligned}$$

Therefore we complete the proof of

THEOREM 3.6. *There are infinitely many free differentiable S^1 -actions on homotopy 13-spheres so that none of them has a characteristic homotopy 9-sphere.*

III. Actions on homotopy 15-spheres.

THEOREM 3.7. *Let S^3 act freely and differentiably on Σ^{15} .*

(i) Σ^{15}/S^3 is a semistandard homotopy quaternion projective 3-space if and only if $I_4(S^3, \Sigma^{15}) = I_6(S^1, \Sigma^{15})$.

(ii) If Σ^{15}/S^3 is a semistandard quaternion projective 3-space, then $I_2(S^1, \Sigma^{15}) = I_6(S^1, \Sigma^{15}) = 2^7 \cdot 217i$ for some $i \in \mathbb{Z}$, and $\Sigma^{15} \in 32bP_{16} \oplus \mathbb{Z}_2$. Hence there are infinitely many free S^3 -actions on some Σ^{15} so that none of them has a S^1 -invariant characteristic 13-sphere S^{13} and S^1 -invariant characteristic homotopy 9-spheres.

Proof. Let $p_i(\Sigma^{15}/S^3) = r_i\alpha^i$, $i = 1, 2, 3$, α the generator of $H^4(\Sigma^{15}/S^3, \mathbb{Z})$. Then

$$I_4(S^3, \Sigma^{15}) = \frac{1}{45} (7r_2 - r_1^2 - 10r_1 + 17) - 1$$

and

$$I_6(S^1, \Sigma^{15}) = \frac{1}{45} (7r_2 - r_1^2 + 5r_1 - 43) - 1,$$

whence $I_4(S^3, \Sigma^{15}) = I_6(S^1, \Sigma^{15})$ if and only if $r_1 = 4$. Repeating the argument used in the proof of Theorem 3.6, since $\sigma(\Sigma^{15}/S^3) = 0$, we see that $r_2 = 31\bar{i} + 12$ and $r_3 = 26\bar{i} + 8$ for some $\bar{i} \in \mathbb{Z}$. But Σ^{15}/S^3 is a spin manifold, hence $\hat{A}_3(\Sigma^{15}/S^3)$ is an even integer and $\hat{A}_3(\Sigma^{15}/S^3) = \bar{i}/(2^6 \cdot 3)$. Thus $\bar{i} = 2^7 \cdot 3 \cdot \bar{j}$ for some $\bar{j} \in \mathbb{Z}$. By Theorem 2.3 (3),

$$\begin{aligned} I_4(S^3, \Sigma^{15}) &= \frac{7}{45} \{ \langle p_2(\Sigma^{15}/S^3), \bar{\chi}'_2 \rangle - \langle p_2(QP^2), \bar{\chi}_2 \rangle \} \\ &= \frac{7}{45} \{ (31\bar{i} + 12) - 12 \} \\ &= \frac{217\bar{i}}{45} = \frac{217 \cdot 2^7 \cdot 3\bar{j}}{45} = 2^7 \cdot 217\bar{j} \quad \text{for some } \bar{j} = 15i \in \mathbb{Z}. \end{aligned}$$

Now let W^{16} be the total space of the disk bundle $D^4 \rightarrow W \rightarrow \Sigma^{15}/S^3$ associated to the principal bundle $S^3 \rightarrow \Sigma^{15} \rightarrow \Sigma^{15}/S^3$. By using standard technique (cf. proof of Lemma 3.2), we obtain the Eells-Kuiper μ -invariant as follows:

$$\begin{aligned} \mu(\Sigma^{15}) &= \frac{12096p_3(W)p_1(W) + 5040p_2(W)^2 - 22680p_2(W)p_1(W)^2 + 9639p_1(W)^4 - 181440}{2^{15} \cdot 3^4 \cdot 5 \cdot 7 \cdot 127} \\ &= \frac{3i(32i-3)}{2 \cdot 127} \pmod{1}. \end{aligned}$$

Hence $\Sigma^{15} \in 32bP_{16} \oplus \mathbb{Z}_2$ by [4].

4. Free S^3 -actions with codimension 4 characteristic homotopy spheres. In this section we like to compare the invariants $I_2(S^1, \Sigma^{4n+3})$ and $I_4(S^3, \Sigma^{4n+3})$ for any free differentiable action of S^3 on a homotopy sphere Σ^{4n+3} , where $S^1 \subset S^3$. The arguments are similar to those used in §2, so we shall use the notation of Theorem 2.3.

THEOREM 4.1. *Suppose that a free differentiable action of S^3 on a homotopy $(4n+3)$ -sphere Σ^{4n+3} is given, $n \geq 3$, and let S^1 be the subgroup of S^3 . Then $I_2(S^1, \Sigma^{4n+3}) = I_4(S^3, \Sigma^{4n+3})$.*

Proof. According to Theorem 2.1, we have

$$\begin{aligned} I_2(S^1, \Sigma^{4n+3}) &= \langle L_n(\tau' \oplus \rho'^{-1}), \chi'_n \rangle - 1 \\ &= \langle L_n(\pi'^* \bar{\tau}' \oplus \pi'^* \eta' \oplus \rho'^{-1}), \chi'_n \rangle - 1 \\ &= \langle L_{n-1}(\pi'^* \bar{\tau}'), \chi'_{n-1} \rangle \langle L_1(\pi'^* \eta' \oplus \rho'^{-1}), \chi'_1 \rangle \\ &\quad + \langle L_{n-2}(\pi'^* \bar{\tau}'), \chi'_{n-2} \rangle \langle L_2(\pi'^* \eta' \oplus \rho'^{-1}), \chi'_2 \rangle \\ &\quad + \cdots + \langle L_n(\pi'^* \eta' \oplus \rho'^{-1}), \chi'_n \rangle + \{ \langle L_n(\pi'^* \bar{\tau}'), \chi'_n \rangle - 1 \} \\ &= \langle L_{n-1}(\bar{\tau}'), \bar{\chi}'_{n-1} \rangle + \langle L_{n-2}(\bar{\tau}'), \bar{\chi}'_{n-2} \rangle \langle L_2(\pi^* \eta \oplus \rho^{-1}), \chi_2 \rangle \\ &\quad + \cdots + \langle L_n(\pi^* \eta \oplus \rho^{-1}), \chi_n \rangle + \{ \langle L_n(\bar{\tau}'), \bar{\chi}'_n \rangle - 1 \} \end{aligned}$$

because $\langle L_1(\pi'^*\eta' \oplus \rho'^{-1}), \chi'_1 \rangle = 1$ and $\pi'^*f^* = f^*\pi^*$. Similarly,

$$\begin{aligned} I_4(S^3, \Sigma^{4n+3}) &= \langle L_{n-1}(\bar{\tau}' \oplus \bar{\rho}'^{-1}), \bar{\chi}'_{n-1} \rangle - \sigma(QP^{n-1}) \\ &= \langle L_{n-1}(\bar{\tau}'), \bar{\chi}'_{n-1} \rangle + \langle L_{n-2}(\bar{\tau}'), \bar{\chi}'_{n-2} \rangle \langle L_1(\bar{\rho}'^{-1}), \bar{\chi}'_1 \rangle \\ &\quad + \cdots + \langle L_{n-1}(\bar{\rho}'^{-1}), \bar{\chi}'_{n-1} \rangle - \sigma(QP^{n-1}). \end{aligned}$$

But $\langle L_n(\pi'^*\bar{\tau}'), \bar{\chi}'_n \rangle - 1 = -\sigma(QP^{n-1})$. Thus it suffices to show that

$$\langle L_k(\pi^*\eta \oplus \rho^{-1}), \chi_k \rangle = \langle L_{k-1}(\bar{\rho}^{-1}), \bar{\chi}_{k-1} \rangle \quad \text{for } k \geq 1$$

since $\langle L_{k-1}(\bar{\rho}'^{-1}), \bar{\chi}'_{k-1} \rangle = \langle L_{k-1}(\bar{\rho}^{-1}), \bar{\chi}_{k-1} \rangle$ for $k \geq 1$. By direct computation, we see easily that the above relation holds for $k=1, 2$. Suppose it is true for $k < m \leq n$. We consider the standard free action of S^1 and S^3 on S^{4m+3} . The same computation gives

$$\begin{aligned} 0 &= I_2(S^1, S^{4m+3}) \\ &= \langle L_{m-1}(\bar{\tau}), \bar{\chi}_{m-1} \rangle + \langle L_{m-2}(\bar{\tau}), \bar{\chi}_{m-2} \rangle \langle L_2(\pi^*\eta \oplus \rho^{-1}), \chi_2 \rangle \\ &\quad + \cdots + \langle L_m(\pi^*\eta \oplus \rho^{-1}), \chi_m \rangle + \{ \langle L_m(\bar{\tau}), \bar{\chi}_m \rangle - 1 \}. \\ 0 &= I_4(S^3, S^{4m+3}) \\ &= \langle L_{m-1}(\bar{\tau}), \bar{\chi}_{m-1} \rangle + \langle L_{m-2}(\bar{\tau}), \bar{\chi}_{m-2} \rangle \langle L_1(\bar{\rho}^{-1}), \bar{\chi}_1 \rangle \\ &\quad + \cdots + \langle L_{m-1}(\bar{\rho}^{-1}), \bar{\chi}_{m-1} \rangle - \sigma(QP^{m-1}). \end{aligned}$$

Again $\langle L_m(\bar{\tau}), \bar{\chi}_m \rangle - 1 = -\sigma(QP^{m-1})$ and so

$$\langle L_m(\pi^*\eta \oplus \rho^{-1}), \chi_m \rangle = \langle L_{m-1}(\bar{\rho}^{-1}), \bar{\chi}_{m-1} \rangle$$

follows from induction hypotheses. This completes the proof of the theorem.

ADDED. In the paper *Differentiable S^1 actions on homotopy spheres* (to appear), G. Brumfiel has found all possible homotopy spheres in dimensions 9, 11 and 13 which admit free differentiable actions of S^1 . He also studied the free differentiable actions of S^1 on homotopy spheres which do not bound π -manifolds.

The authors were informed that the action on homotopy spheres not in bP_{2n} were also studied by D. Frank.

Concerning the problem related to the existence of characteristic homotopy spheres, the authors also showed the following results: There are homotopy $(4n+1)$ - or $(4n+3)$ -spheres which admit infinitely many topologically distinct free differentiable actions of S^1 or S^3 with characteristic homotopy spheres in certain dimensions and without characteristic homotopy spheres in some other dimensions (cf. *Free differentiable actions of S^1 and S^3 on homotopy spheres*, Proc. Amer. Math. Soc. **25** (1970), 864-869). We remark that Theorem 4.1 can be generalized as follows with the same proof:

$$I_{4k+2}(S^1, \Sigma^{4n+3}) = I_{4k}(S^3, \Sigma^{4n+3}) + I_{4k+4}(S^3, \Sigma^{4n+3}) \quad \text{for } 0 \leq k < n-2.$$

See also a paper of B. Conrad (*Extending free circle actions on spheres to S^3 actions*, mimeographed, Temple University, Philadelphia, Pa., 1970).

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